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# *Hlawka-Mück techniques for option pricing*

## Quasi-Monte Carlo methods with NIG distribution

joint work with J. Hartinger and M. Predota

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# Overview

- Sample problem: Valuing Asian options
- Crude Monte Carlo simulation
- Quasi-Monte Carlo estimators
  - Integral transformation
  - Ratio of uniforms
  - Hlawka-Mück's method for density $f^Q$
- Numerical comparison

# Sample problem: Valuing Asian options

arithmetic mean until expiration time

## Pay-Off (discrete Asian option, call)

$$P(S_T) = \left( \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+$$

$(S_t)_{t \geq 0}$ ... price process, $K$ ... strike price

$S_t = e^{X_t}$ with Levy process $(X_t)_{t \geq 0}$

Increments $h_i = X_i - X_{i-1}$ with distribution $H$
(e.g. NIG, Variance-Gamma, Hyperbolic, ...)

### NIG distribution

Use the NIG distribution for the increments $h_i \sim H^Q$.
Advantage: closed under convolution $\Rightarrow$ dimension reduction, sample only weekly instead of daily

## Valuation

Using fundamental theorem (Schachermayer):

$$C_{t_0} := e^{-r(t_n - t_0)} \mathbb{E}^Q \left[ \left( \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ \right]$$

$r$ ... constant interest rate
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# Crude Monte Carlo simulation

Direct simulation of the process, arithmetic mean over $L$ pathes:

1. Simulate $L$ price pathes $\left((S_0^{(l)}, S_1^{(l)}, \dots)\right)_{l \geq 1}$
   with $S_i = e^{X_i}$, $X_i = X_{i-1} + h_i$, $h_i \overset{i.i.d}{\sim} H^Q$.
2. Calculate pay-off $P^{(l)}$ for each path $l$
3. Crude MC estimator: $\hat{C}_0 = e^{-r(t_n - t_0)} \frac{1}{L} \sum_{l=1}^{L} P^{(l)}$

Random numbers $h_i \sim H^Q$ created using acceptance-rejection.

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1. Hlawka-Mück method for direct creation of $(x_n)_{1 \leq n \leq N} \overset{i.i.d.}{\sim} NIG$
   $\Rightarrow$ direct QMC calculation of the expectation value

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1. Hlawka-Mück method for direct creation of $(x_n)_{1 \leq n \leq N} \stackrel{i.i.d.}{\sim} NIG$ $\Rightarrow$ direct QMC calculation of the expectation value
2. Transformation of the integral using a suitable density (Ratio of uniforms, "Hat") $\Rightarrow$ variance reduction (if done right)

# Transformation

Using a distribution $K(\vec{x}) = u$:

$$\int_{\mathbb{R}^n} P(\vec{x}) f_H^Q(\vec{x}) d\vec{x} = \int_{[0,1]^n} P(K^{-1}(u)) \frac{f_H^Q(K^{-1}(u))}{k(K^{-1}(u))} du$$

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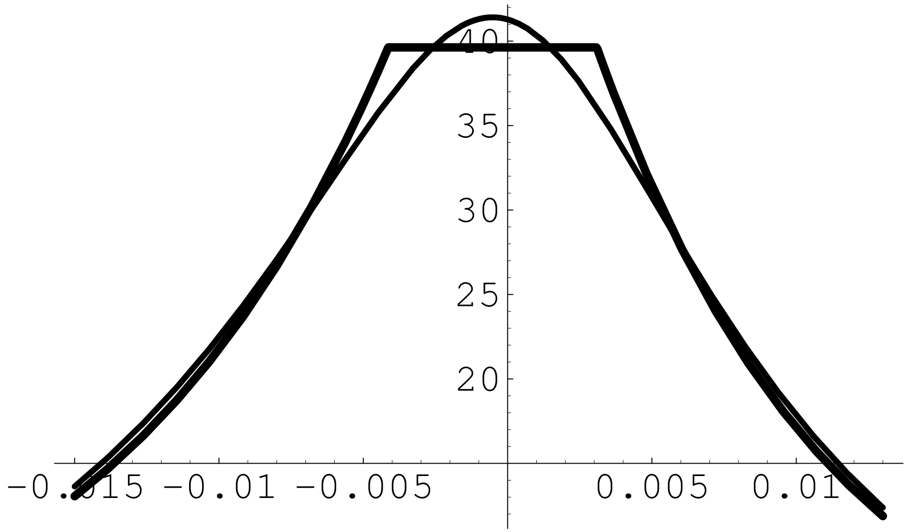
Problem: "Usual" transformation $F(x) = u$ leads to integrand with unbound variation

# Ratio of uniforms

"Hat" function, good choice for integral transformation



# Hlawka-Mück

- Hlawka and Mück (1972): transformation of uniformly distributed sequences to low-discrepancy sequences with density $\rho$
- Hlawka (1997): simpler construction for densities $\rho = \rho_1(x_1)\rho_2(x_2)\ldots\rho_s(x_s)$

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For density $\rho(x) = \rho_1(x_1) \cdot \dots \cdot \rho_s(x_s)$ define

$$F_1(x^{(1)}) = \int_0^{x^{(1)}} \int_0^1 \dots \int_0^1 \rho(u_1, \dots, u_s) du_1 \dots du_s$$

$$\vdots$$

$$F_s(x^{(s)}) = \int_0^1 \int_0^1 \dots \int_0^{x^{(s)}} \rho(u_1, \dots, u_s) du_1 \dots du_s$$

Creation of net $(y_k)_{1 \le k \le N}$ with density $\rho$:

$$y_k^{(j)} = \frac{1}{N} \sum_{r=1}^{N} \left\lfloor 1 + x_k^{(j)} - F_j(x_r^{(j)}) \right\rfloor, \qquad j = 1, \dots s, \qquad k = 1, \dots N$$

## Discrepancy

The discrepancy of $(y_k)_{1\leq k\leq N}$ can be bounded by

$$D_N\left((y_k),\rho\right) \leq (2+6sM(\rho))D_N\left((y_k)\right)$$

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## QMC estimator

1. Creation of low-discrepancy points with density $\frac{f_H^Q(K^{-1}(x))}{k(K^{-1}(x))}$
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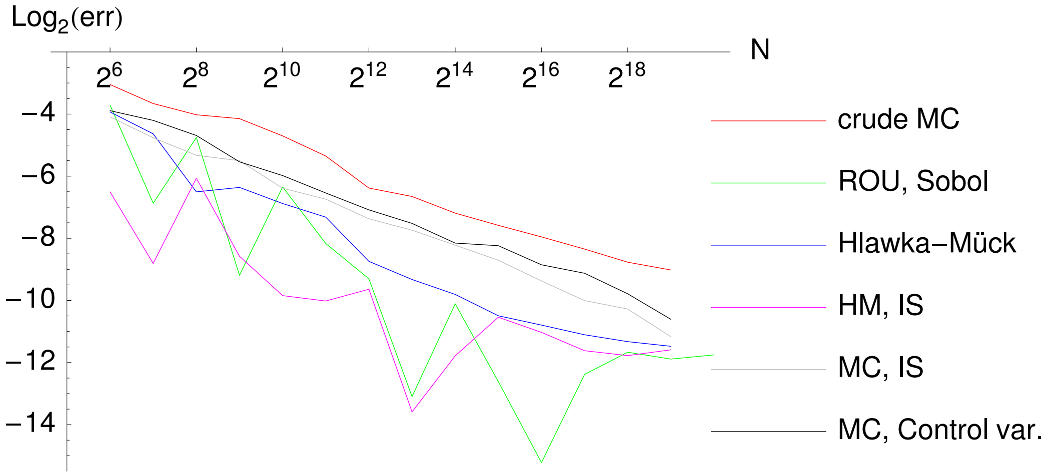
2. Transformation $[0,1]^n$ to $\mathbb{R}^n$ of the sequence using double-exponential distribution $K^{-1}(x)$.

Estimator similar to crude Monte Carlo

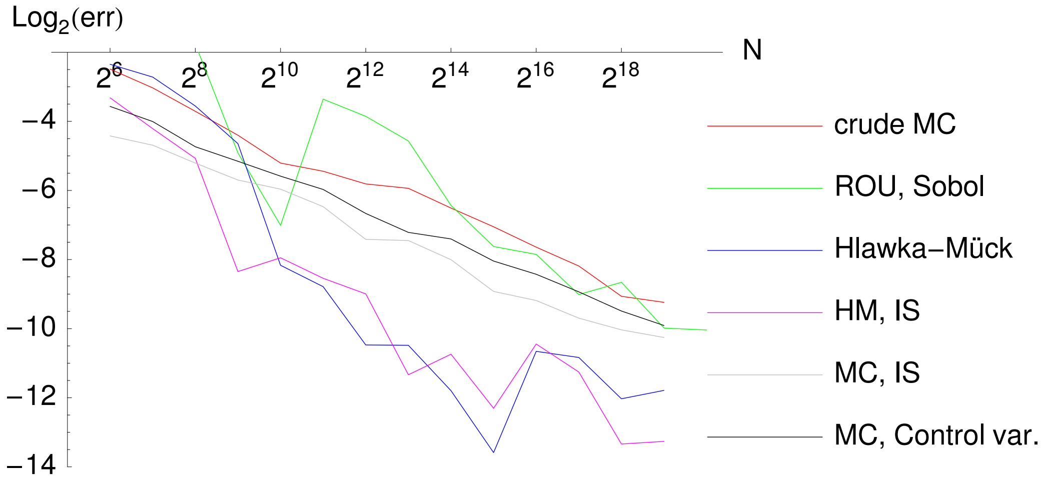
# Numerical results

## Dimension 4




ROU and Hlawka-Mück are considerably better than Monte Carlo and control variate

## Dimension 12




- ROU looses performance
- Hlawka-Mück gives best results