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QMC Methods for the solution of delay differential equations[a]

In this paper we will discuss the application of so-called Runge Kutta Quasi-Monte Carlo (RKQMC) methods (as proposed by Lécot, Koudiraty, Coulibaly, and Stengle) to heavily oscillating differential equations. The delayed argument is approximated by Hermite interpolation which transforms the equation into an ordinary differential equation so that the RKQMC methods can be applied. We will give a short discussion of this method and its advantages as well as its drawbacks, and give some more numerical results.

1. The problem

In this paper we will be concerned with the numerical solution of delay differential equations of the form

$$
\begin{aligned}
y'(t) &= f\left(t, y(t), y(t - \tau(t))\right), \quad \text{for } t \geq t_0, \\
y(t) &= \phi(t), \text{ for } t \leq t_0 .
\end{aligned}
\tag{1}
$$

Here $\tau(t)$ is the continuous, strictly positive delay function satisfying $t_1 - \tau(t_1) \leq t_2 - \tau(t_2)$ for $t_1 \leq t_2$, and $\phi(t)$ is the piecewise continuous initial function on the interval $[t_0 - \tau(t_0), t_0)$. Although the method can be applied to any delay differential equation, only for heavily oscillating ones (or equations with heavily oscillating solutions) it can significantly improve the error in comparison to conventional Runge Kutta methods.

Such delay differential equations appear frequently in all engineering and scientific fields when one tries to describe a process observed in nature. In most cases, a process does not depend on just one retarded argument, but usually on all past values by an integration with a certain kernel that weights some times greater than others. Instead of mathematically treating such a delay integro-differential equation, one then usually replaces the dependence on all past values by just the value of the most important past values. One may question the validity of such a replacement in some cases, but there are many processes where the peaks in the kernel are so distinguished that the kernel is virtually a sum of delta functionals. In particular, when describing heavily oscillating electronic circuits, the speed of light is still so much larger than the oscillation, that all fluctuations (quantum fluctuations or material effects) can safely be ignored.

As described in detail in [2], we can obtain approximate numerical solutions y_n at times $t_n = t_{n-1} + h_{n-1}$, $n \geq 0$, to equation (1) by combining the Hermite interpolation method for delay differential equations (see e.g. [7]) with the stochastic Runge Kutta Monte Carlo ([8]) and Quasi-Monte Carlo methods ([5], [1], [4]). The values of y_n will be calculated sequentially, so that for the calculation of y_n the history y_k, $0 \leq k < n$ is available. This allows to interpolate the retarded argument $y(t - \tau(t))$ from the already calculated values y_k at times $t_k < t$, so that the delay differential equation can be approximated by an ordinary differential equation. The DDE explicitly gives the values of $y'(t_n)$, and thus we will use Hermite interpolation for the retarded values.

Taylor-expansion of equation (1) in $y(t)$ and conventional Quasi-Monte Carlo integration in t leads to s-order schemes of the form

$$
y_{n+1} = y_n + \frac{h_n}{s! N} \sum_{0 \leq j < N} G_s \left(\bar{t}_{j,n}; y_n, z_{(y_i)_{i \leq n}}(\bar{t}_{j,n}) \right),
\tag{2}
$$

where $z_{(y_i)_{i \leq n}}$ is the interpolation function used for the numerical experiments. Following the lines of Lécot [5] and Koudiraty [4], the corresponding functions G_s for $s = 1, 2$ are obtained as:

$$
G_1\left(u; y; z(\cdot)\right) = f\left(u, y, z(u - \tau(u))\right)
\tag{3}
$$

$$
G_2\left(\bar{u}; y; z(\cdot)\right) = f\left(\bar{u}_1, y, z(\bar{u}_1 - \tau(\bar{u}_1))\right) + \frac{1}{\beta} f\left(\bar{u}_2, y, z(\bar{u}_2 - \tau(\bar{u}_2))\right) +
\tag{4}
$$

$$
+ \frac{1}{\alpha} f\left(\bar{u}_2, y + \alpha h_n f\left(\bar{u}_1, y, \bar{u}_1 - \tau(\bar{u}_1)\right), z(\bar{u}_2 - \tau(\bar{u}_2))\right)
$$

with the condition $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ ([5]). The lenghty expression for $G_3\left(\bar{u}; y; z(\cdot)\right)$ can be found in [4] for ODE or in our paper [2], where we also discuss the schemes for DDE in more detail and give a general convergence proof. Previous results on RKQMC methods for ODE were obtained by Lécot ([5]), and Lécot and Koudiraty ([4]), solution methods for DDE using Hermite interpolation were investigated e.g. in Oberle and Pesch [7], and a good overview on Quasi-Monte Carlo methods can be found in [6].

In contrast to conventional Runge Kutta methods, where only up to five or six values of $f(y, t)$ are used to calculate y_{n+1} from y_n, these Runge Kutta Quasi-Monte Carlo methods try to minimize the error for heavily oscillating equations by a Quasi-Monte Carlo integration over the whole interval $[t_n, t_{n+1})$. This integration leads to N function evaluations for each step and ist thus more expensive then the conventional methods. However, these function evaluations do not depend on each other and can thus be calculated in parallel. Additionally, the RKQMC methods can be applied with a much larger step size then the conventional methods. For our example, we used $h = 0.001$ for Runge Kutta schemes, and $h = 0.01$ for RKQMC methods. In [3] we presented an even more striking numerical results, when we investigated the differential equation of two retarded arguments

$$y'(t) = \pi \frac{\lambda}{2} \left(y\left(t - 2 - \frac{3}{2\lambda}\right) - y\left(t - 2 - \frac{1}{2\lambda}\right)\right), \quad t \geq 0 \tag{5}$$
$$y(t) = sin(\lambda t \pi), \quad t < 0 \, ,$$

with the exact solution $y(t) = sin(\lambda t\pi)$. Since the k-th derivative of $y(t)$ is only bounded by λ^k, the interpolation error will also grow with λ^k, and thus for large λ the method is expected to become unstable. As the figure below shows, for conventional Runge-Kutta schemes this happens for values around 2^7 to 2^{10}, whereas RKQMC methods stay stable until $\lambda = 2^{13}$, and even then display a much smaller error $S^{\lambda,(meth)}$.
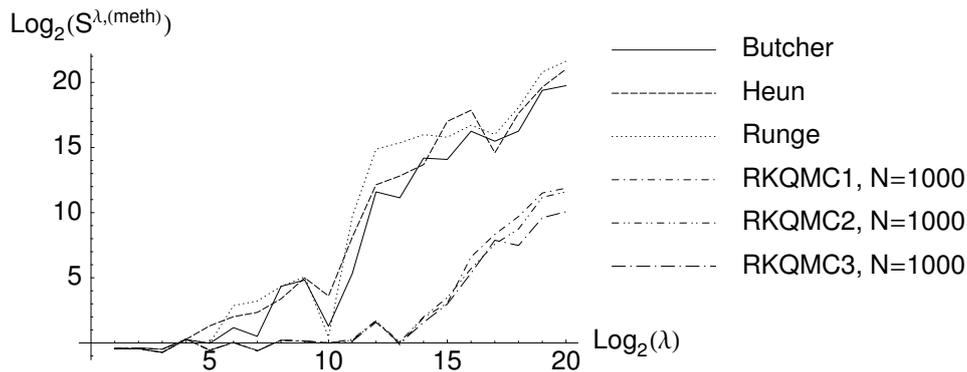
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