Runge-Kutta quasi-Monte Carlo methods for delay differential equations[1]

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In this paper, we consider initial value problems for delay (retarded) differential equations (DDE) having the form

$$y'(t) = f\left(t, y(t), y(t-\tau(t))\right), \quad \text{for } t \geq t_0, \tag{1}$$
$$y(t) = \phi(t), \text{ for } t \leq t_0 , \tag{2}$$

where $y(t)$ is a d-dimensional real-valued function, $\tau(t)$ is the delay function, which satisfies $t_1 - \tau(t_1) \leq t_2 - \tau(t_2)$ for $t_1 \leq t_2$. Furthermore, $\phi(t)$ is the initial function, which is assumed to be piecewise continuous on the interval $(\inf_{t_0 \leq t}(t-\tau(t)), t_0)$.

Runge Kutta methods for such differential equations are well known and use mostly interpolation of the retarded values from the past solution values already calculated.

Stengle [5] proposed a randomized Runge Kutta scheme for ordinary differential equations which vary significantly faster in t than in $y(t)$, which leads to a significant improvement of the error. Lecot [3], Coulibaly [1] and Koudiraty [2] generalized this method and use low-discrepancy sequences for the numerical integration. This allows them to give an explicit upper bound for the error of the solution at time step t_n.

In our work, we combine these two methods and apply Runge-Kutta quasi-Monte Carlo (RKQMC) methods to delay differential equations. This leads to schemes of the form

$$y_{n+1} = y_n + \frac{h_n}{s!N} \sum_{0 \leq j < N} G_s\left(t_{n,j}, y(t_n); (y_i)_{i \leq n}\right) , \tag{3}$$

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where $G_s(t, y(t); (y_i)_{i\leq n})$ is the differential increment function of the RKQMC scheme of order s with the retarded values interpolated from $(y_i)_{i\leq n}$ by Hermite interpolation and the $t_{n,j}$ are uniform (quasi-)random variates in the interval $(t_n, t_{n+1})^s$. We prove in a general framework that like in the conventional Runge Kutta approach to delay differential equations, the error of the method is of order $\min(s, q)$, where q is the order of the interpolation method and s is the order of the Runge Kutta method, if a low-discrepancy point set (e.g. Halton's, Sobol's sequence, or a (t, s)-sequence as defined by Niederreiter) with $N = \mathcal{O}(h^{-\min(s,q)})$ elements is used.

For out computational experiments, we extend the first-, second- and third-order RKQMC schemes proposed by Lecot, Coulibaly and Koudiraty and numerically investigate their application to some delay differential equations. Here it shows, that the RKQMC schemes for DDE really outperform even high-order Runge-Kutta schemes, if $f(t, y(t), y'(t))$ oscillates very fast.

[1] I. Coulibaly and C. Lécot. A quasi-randomized Runge-Kutta method. *Mathematics of Computation*, 68(226):651–659, 1999.

[2] A. A. Koudiraty. *Numerical analysis of some quasi-Monte Carlo Methods.* PhD thesis, Université de Savoie, 2001.

[3] C. Lécot. Quasi-randomized numerical methods for systems with coefficients of bounded variation. *Mathematics and Computers in Simulation*, 55:113–121, 2001.

[4] H. J. Oberle and H. J. Pesch. Numerical Treatment of Delay Differential Equations by Hermite Interpolation. *Numer. Math.*, 37:235–255, 1981.

[5] G. Stengle. Error analysis of a randomized numerical method. *Numer. Math.*, 70:119–128, 1995.