Transformation methods for the creation of non-uniformly distributed low-discrepancy sequences

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Motivation

Many problems from finance and engineering need sequences that follow a distribution other than the uniform distribution.
How can we generate them for QMC?

Methods known from Monte Carlo:

- Inversion method, using the quantile function of the distribution. Preserves the discrepancy.
- Acceptance-Rejection method
- Smoothing method:
  Adaption of acceptance-rejection for QMC (Wang)
  Integrand with A-R is changed/modified to perform better, integral value stays the same

Problems with conventional methods for QMC

- Inversion:
  Problem: Usually the quantile function is not explicitely known.
- Acceptance-Rejection: unsuited for QMC
  - introduces jumps in the integrand function $\Rightarrow$ unbounded variation
  - lots of points are thrown away in high dimensions
- Smoothing:
  Problem: Only applicable for integration problems, does not generate a sequence. For path simulation the h-distributed sequences are explicitely needed

Basic definitions

- Star-discrepancy of a sequence:

$$D_N\left((\mathbf{x}_i)_{i\in\mathbb{N}}\right) = \sup_{J=[0,\mathbf{z})\subset[0,1)^s} \left| \frac{\#(\mathbf{x}_i, J)}{N} - \lambda_s(J) \right|$$

- Low-discrepancy sequence: $D_N((\mathbf{x}_i)) = \mathcal{O}\left(\frac{\log^s N}{N}\right)$
- h-Discrepancy of sequence $(\mathbf{x}_i)_{i\in\mathbb{N}}$:

$$D_{h,N}\left((\mathbf{x}_i)_{i\in\mathbb{N}}\right) = \sup_{J=[0,\mathbf{z})\subset[0,1)^s} \left| \frac{\#(\mathbf{x}_i, J))}{N} - H(J) \right|$$

- L^p-Discrepancy of sequence $(x_i)_{i\in\mathbb{N}}$:

$$D^{(p)}\left((\mathbf{x}_i)_{i\in\mathbb{N}}\right) = \left(\int_{[0,1]^s} \left(\frac{\#(\mathbf{x}_i, [0,\mathbf{z}))}{N} - \lambda_s([0,\mathbf{z}))\right)^p d\mathbf{z}\right)^{\frac{1}{p}}$$

Hlawka-Mück method

In 1972 Hlawka and Mück propose a method to generate h-distributed (double) sequences.

For $f(\mathbf{x}) = f_1(x_1) \cdot \ldots \cdot f_s(x_s)$ simpler construction (Hlawka, '98):

Let $\omega = (x_i)_{i \in \mathbb{N}} \sim U(0,1)$ with discrepancy $D_N(\omega)$, and $f(x_1, \ldots, x_s)$ an s-dimensional density function. Define $M_f = \sup f(\mathbf{x})$. Then $\tilde{\omega}$ with

$$\tilde{y}_{k,j} = \frac{1}{N} \sum_{r=1}^{N} \lfloor 1 + x_{k,j} - F_j\left(x_{r,j}\right) \rfloor = \frac{1}{N} \sum_{r=1}^{N} \chi_{[0, x_{k,j}]}\left(F_j\left(x_{r,j}\right)\right)$$

has an F-discrepancy of

$$D_{N,F}(\tilde{\omega}) \leq (1 + 3mM_f) D_N(\omega) \; .$$

Problems of the H-M method

The Hlawka-Mück method has two serious practical problems:

- Effort: $\mathcal{O}(N^2)$ for generation
  Each y_k need the evaluation of $\sum_{r=1}^{N} \ldots$.
  -) The order cannot be easily reduced
  The evaluation of $F(x_i)$ can be done before.
- If you change N, the whole point set needs to be regenerated.
- Transformed sequence contains identical points
  All points are transformed to the grid $\frac{k}{N}$, $0 \leq k \leq N$.
  -) Can be solved, e.g. by our adaption, by using a continuous approximation to $F(x)$ instead of the step function.

An adaption of the Hlawka-Mück method

Inversion, approximate distribution function F by continuous, piecewise linear function.

One-dimensional: Let $\omega = (x_i)_{i\in\mathbb{N}}$ a uniformly distributed sequence. Define $x_k^- = \max_{x_i}(F(x_i) < x_k)$ and $x_k^+ = \min_{x_i}(F(x_i) > x_k)$. Then for the the sequence generated by

$$y_k = \frac{F(x^+) - x_k}{F(x^+) - F(x^-)} x^- + \frac{x_k - F(x^-)}{F(x^+) - F(x^-)} x^+$$

it holds that $|y_k - F^{-1}(x_k)| < D_N(\omega)$ and $|F(y_k) - x_k| < M D_N(\omega)$. Consequently, $D_N(\tilde{\omega}) \leq (1+M) D_N(\omega)$.

For s-dimensional distributions with $f = f_1 \ldots f_s$ the bound is $D_N(\tilde{\omega}) \leq (1+2M)^s D_N(\omega)$.

Other possible modifications (approx., support pts)

- Why do we use the sequence to approximate $F(x)$?
  Quality of approximation needs to increase with N.
- Why don't we use $\frac{k}{N}$ as support points for the empirical distribution function?
  Problem: If we change N, all support points need to be recalculated. With the H-M method (and our adaption), only the sum needs to be redone, not the $F(x_k)$.
  Also, no significant improvement in order of discrepancy (just a constant).
- Why not use the same support points for all dimensions?
  Quality of approximation and of the sequence decoupled.
  Discrepancy bound includes both discrepancies, asymptotics unchanged.

$L^{(2)}$ discrepancy of transformed sequences

Integration error can also be bounded by the $L^{(2)}$ discrepancy (Sloan, Wozniakowski):

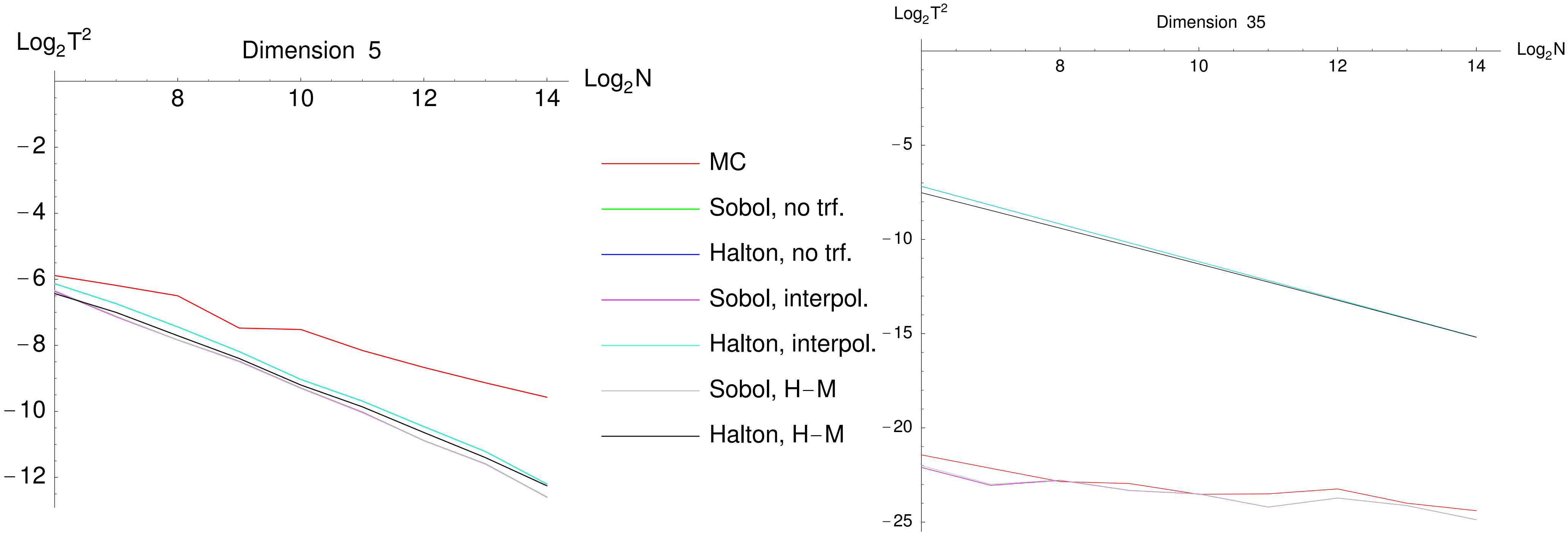
$$|I_d(f) - Q_{n,d}(f)| \leq disc^{(2)}((x_i))\,\|f\|_d^2$$

with $disc^{(2)} = \left(\sum_{u \subset D} (T_u^{(2)})^2\right)^{1/2}$ and $L^{(2)}$-variation $\|f\|_d^2$.

$T^{(2)}$ can be explicitely calculated:

$$\left(T^{(2)}\right)^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \prod_{k=1}^{d} \left(1 - \max\left(x_{ik}, x_{jk}\right)\right) - \frac{2^{1-d}}{N} \sum_{i=1}^{N} \prod_{k=1}^{d} \left(1 - x_{ik}^2\right) - \frac{1}{3^d}$$

Results: $L^{(2)}$ discrepancy



- Halton looses quality for high dimensions
- MC and Sobol have the same $L^{(2)}$ discrepancy in 50 dim
- H-M and our transformations do not affect discrepancy

Example: Asian options (50 dim)

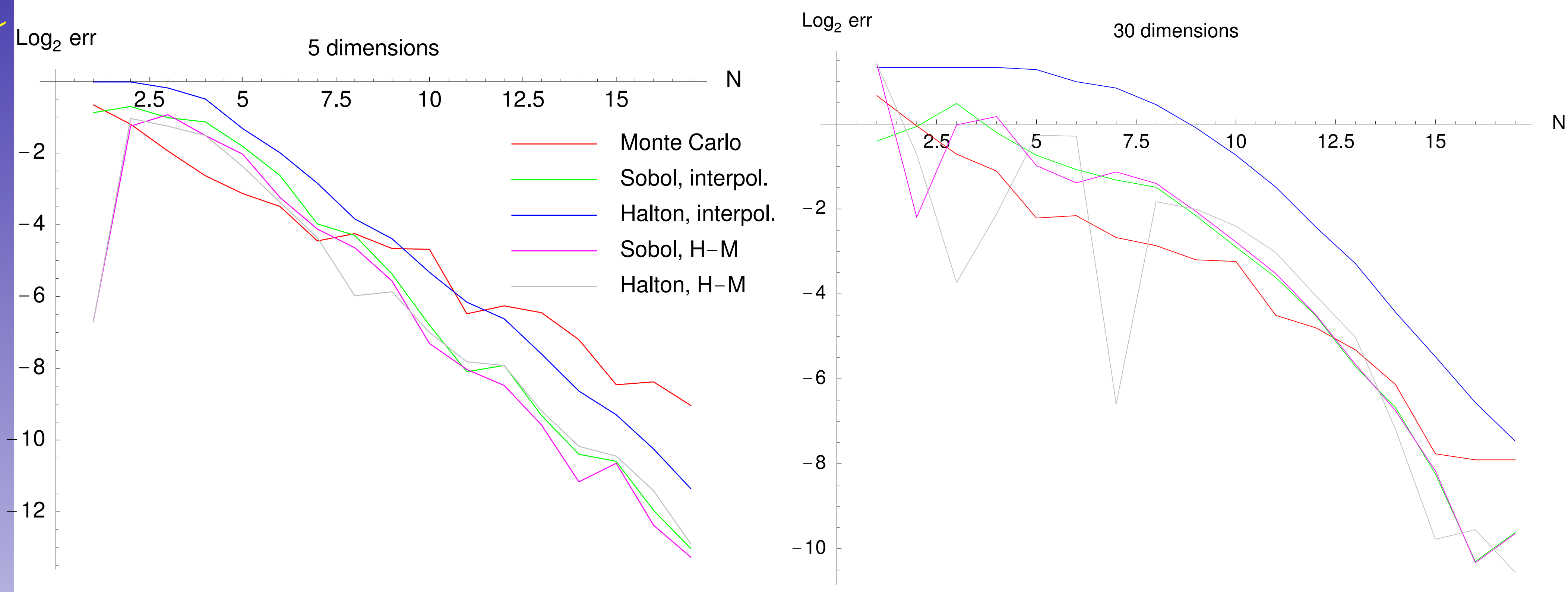
Asian option with payoff

$$P = \mathbb{E}_{NIG}\left(e^{-rT}\left(S_0 \sum_{i=1}^{d} \prod_{j=1}^{i} e^{x_i} - K\right)^+\right)$$

When using inversion, integrand is unbounded ($V(f) = \infty$)!

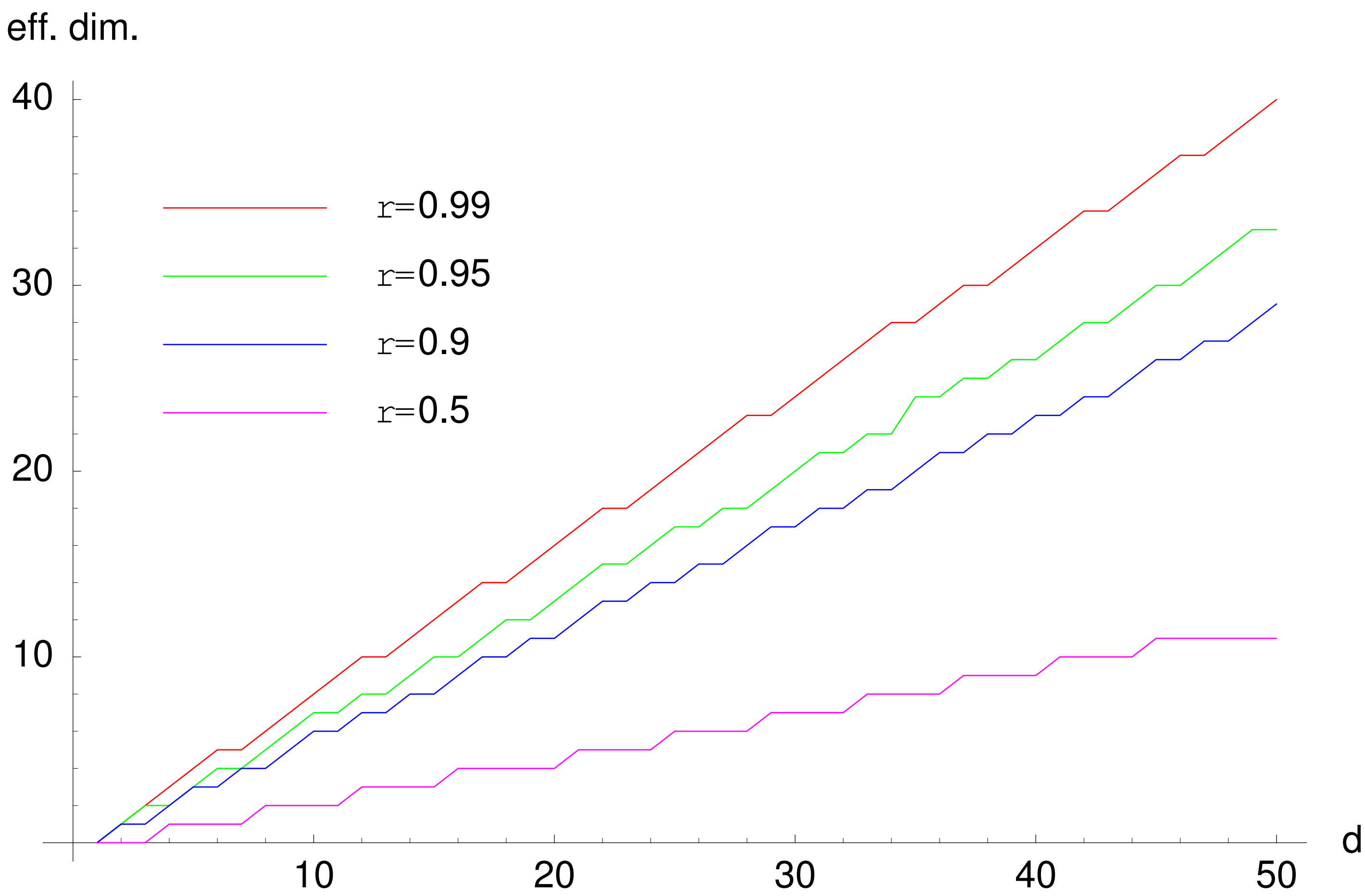
- Theorems by Sobol, and Hartinger/K./Tichy and Hartinger/K./Predota prove convergence.
- Only slightly worse convergence order than Koksma-Hlawka for bounded variation.

Asian options (50 dim): results



- QMC loose quality for high dimensions
- Halton with interpolation doesn't work
- Sobol works both with H-M and its interpolation adaption

Problem: high effective dimension



high effective dimension (e.g. d=50, eff. dim=41), no dimension reduction techniques (like BB, PCA) known for Asian options with NIG distribution