

# RISK THEORY WITH A NON-LINEAR DIVIDEND BARRIER\*

HANSJÖRG ALBRECHER AND REINHOLD KAINHOFER

## Abstract

In the framework of classical risk theory we investigate a surplus process in the presence of a non-linear dividend barrier and derive equations for two characteristics of such a process, the probability of survival and the expected sum of discounted dividend payments. Number-theoretic solution techniques are developed for approximating these quantities and numerical illustrations are given for exponential claim sizes and a parabolic dividend barrier.

## 1 Introduction

Let us consider the classical risk process  $R_t = u + ct - \sum_{i=1}^{N(t)} X_i$ , where  $c$  is a constant premium intensity,  $N(t)$  denotes a homogeneous Poisson process with intensity  $\lambda$  which counts the claims up to time  $t$  and the claim amounts  $X_i$  are iid random variables with distribution function  $F(y)$ . In this context  $R_t$  represents the surplus of an insurance portfolio at time  $t$  (for an introduction to classical risk theory see for instance GERBER [13] and THORIN [24] or more recently ASMUSSEN [4]). As usual we assume  $\mu = E(X_i) < \infty$  and  $c > \lambda \int_0^\infty y dF(y)$ . A reasonable modification of this model is the introduction of a dividend barrier  $b_t$ , i.e. whenever the surplus  $R_t$  reaches  $b_t$ , dividends are paid out to the shareholders with intensity  $c - db_t$  and the surplus remains on the barrier, until the next claim occurs. This means that the risk process develops according to

$$dR_t = c dt - dS_t \quad \text{if } R_t < b_t \quad (1)$$

$$dR_t = db_t - dS_t \quad \text{if } R_t = b_t, \quad (2)$$

where we have used the abbreviation  $S_t = \sum_{i=1}^{N(t)} X_i$ . Together with the initial capital  $R_0 = u$ ,  $0 \leq u < b_0 < \infty$ , this determines the risk process  $\{R_t, t \geq 0\}$  (cf. Figure 1).

Two quantities of particular interest in this context are the probability of survival  $\phi(u, b) = Pr(R_t \geq 0 \forall t > 0 | R_0 = u, b_0 = b)$  (or alternatively the probability of ruin  $\psi(u, b) = 1 - \phi(u, b)$ ) and the expected sum of discounted dividend payments  $W(u, b)$ .

Dividend barrier models have a long history in risk theory (see e.g. [9, 7, 13]). For a survey on the relation between dividend payments and tax regulations we refer to [3, 5]. GERBER [12] showed that barrier dividends constitute a complete family of Pareto-optimal dividends. In the case of a horizontal dividend barrier  $b_t \equiv b_c = \text{const.}$ , it is easy to see that  $\phi(u, b) = 0 \forall 0 \leq u \leq b$ . SEGERDAHL [21] used the technique of integro-differential equations to derive the characteristic function of the time to ruin in the presence of a

---

\*This research was supported by the Austrian Science Foundation Project S-8308-MAT

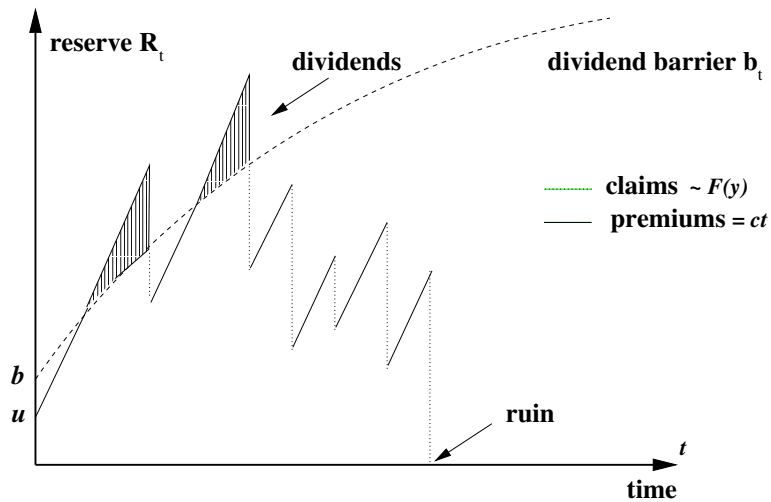


FIGURE 1: A typical sample path of  $R_t$

horizontal dividend barrier for exponentially distributed claims. This approach was generalized by GERBER AND SHIU [15]. PAULSEN AND GJESSING [19] calculated the optimal value of  $b_c$  that maximizes the expected value of the discounted dividend payments in an economic environment. Recently IRBÄCK [17] developed an asymptotic theory for a high horizontal dividend barrier.

If one allows for monotonically increasing  $b_t$  in the model, a positive probability of survival can be achieved. The case of linear dividend barriers is fairly well understood: GERBER [11] derived an upper bound for the probability of ruin for  $b_t = b + at$  by martingale methods and in [14] he obtained exact solutions for the probability of ruin and the expected sum of discounted dividend payments  $W(u, b)$  in terms of infinite series in the case of exponentially distributed claim amounts. This result was generalized to arbitrary Erlang claim amount distributions in SIEGL AND TICHY [22] by developing a suitable solution algorithm. The convergence of this algorithm was proved by ALBRECHER AND TICHY [1].

Apart from mathematical simplicity there is no compelling reason to restrict the model to linear dividend barriers. Moreover, simulations indicate that by choosing an appropriate dividend barrier, the expected value of discounted dividend payments  $W(u, b)$  can be increased, while the probability of survival  $\phi(u, b)$  stays constant (cf. ALEGRE ET AL. [2]).

In this paper non-linear dividend barrier models are investigated. In Section 2 we derive integro-differential equations for  $\phi(u, b)$  and  $W(u, b)$  and discuss the existence and uniqueness of the corresponding solutions. Our main focus is on the development of efficient numerical algorithms to obtain those quantities. More precisely, we adapt number-theoretic solution methods in the spirit of [25] to the current situation (Section 3). Finally Section 4 gives numerical results for the special case of a parabolic dividend barrier and exponential claim amount distributions.

## 2 The Model

**Model A:** We consider a classical risk process extended by a dividend barrier of type

$$b_t = \left( b^m + \frac{t}{\alpha} \right)^{1/m} \quad (\alpha, b > 0, m > 1).$$

Note that  $m = 1$  corresponds to the linear barrier case.

The probability of survival  $\phi(u, b)$  can then be expressed as a boundary value problem in the following way: Conditioning on the occurrence of the first claim, we get for  $u < b$

$$\begin{aligned} \phi(u, b) = & (1 - \lambda dt) \phi \left( u + c dt, \left( b^m + \frac{dt}{\alpha} \right)^{1/m} \right) + \\ & + \lambda dt \int_0^{u+c dt} \phi \left( u + c dt - z, \left( b^m + \frac{dt}{\alpha} \right)^{1/m} \right) dF(z). \end{aligned} \quad (3)$$

Taylor series expansion of the functions  $\phi$  on the right-hand side of (3) and division by  $dt$  shows that  $\phi$  satisfies the equation

$$c \frac{\partial \phi}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial \phi}{\partial b} - \lambda \phi + \lambda \int_0^u \phi(u - z, b) dF(z) = 0, \quad (4)$$

which, for reasons of continuity, is valid for  $0 \leq u \leq b$ . For  $u = b$  we get along the same line of arguments

$$\begin{aligned} \phi(u, b) = & (1 - \lambda dt) \phi \left( \left( b^m + \frac{dt}{\alpha} \right)^{1/m}, \left( b^m + \frac{dt}{\alpha} \right)^{1/m} \right) + \\ & + \lambda dt \int_0^{\left( b^m + \frac{dt}{\alpha} \right)^{1/m}} \phi \left( \left( b^m + \frac{dt}{\alpha} \right)^{1/m} - z, \left( b^m + \frac{dt}{\alpha} \right)^{1/m} \right) dF(z), \end{aligned} \quad (5)$$

from which it follows that

$$\frac{1}{\alpha m b^{m-1}} \frac{\partial \phi}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial \phi}{\partial b} - \lambda \phi + \lambda \int_0^u \phi(u - z, b) dF(z) = 0. \quad (6)$$

Comparing (4) and (6) we thus obtain the boundary condition

$$\frac{\partial \phi}{\partial u} \Big|_{u=b} = 0. \quad (7)$$

A further natural requirement is

$$\lim_{b \rightarrow \infty} \phi(u, b) = \phi(u), \quad (8)$$

where  $\phi(u)$  is the probability of survival in absence of the barrier.

Contrary to ruin, the crossing of the dividend barrier is a much desired event. For equal slopes of the barrier at time 0, the expected time until the first crossing of the dividend

barrier will be considerably less for sub-linear barriers as introduced above than for the linear case. A quantitative result in this direction follows from BOOGAERT ET AL. [6] who used a martingale technique to derive upper bounds for the probability  $Pr(D > t)$  that the surplus process does not reach a given barrier before time  $t$ . Adapting these results to our situation, we obtain

$$Pr(D > t) \leq \frac{\lambda \mu t}{u - (b^m + t/\alpha)^{1/m} + ct}$$

for all  $t$  that satisfy  $u + ct > (b^m + t/\alpha)^{1/m}$ .

Let furthermore  $W(u, b)$  denote the expected present value of the future dividend payments, which are discounted with a constant intensity  $\delta$ , and stop when ruin occurs. Then, in a similar way to (3) and (5), one can derive the integro-differential equation

$$c \frac{\partial W}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial W}{\partial b} - (\delta + \lambda) W + \lambda \int_0^u W(u - z, b) dF(z) = 0, \quad (9)$$

with boundary condition

$$\left. \frac{\partial W}{\partial u} \right|_{u=b} = 1. \quad (10)$$

In the actuarial literature [11, 26] there has been some interest in models where dividends can also be paid after a ruin event (this makes sense since ruin of a portfolio is a technical term used in decision making and does not necessarily imply bankruptcy). If we allow for dividend payments after ruin in our model, then along the same line of arguments as above, we obtain the following equation for the expected value  $V(u, b)$  of the discounted dividend payments

$$c \frac{\partial V}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial V}{\partial b} - (\delta + \lambda) V + \lambda \int_0^\infty V(u - z, b) dF(z) = 0, \quad (11)$$

and the initial condition  $\left. \frac{\partial V}{\partial u} \right|_{u=b} = 1$ . Note that for a linear dividend barrier the corresponding integro-differential equation was much simpler, because  $V$  could be expressed as a function of one variable only (cf. [26]); for a non-linear barrier this is no longer the case.

**Model B:** In addition to Model A, we will also consider a “finite-horizon” version of the model, namely we introduce an absorbing upper barrier  $b_{max} \equiv const$ . If the surplus process  $R_t \geq b_{max}$  for some  $t > 0$ , it is absorbed and the company is considered to have survived. From an economic point of view this can be interpreted that the company will then decide to pursue other forms of investment strategies. Mathematically, this model has some nice features (e.g. the process stops in finite time with probability 1). The boundary value problem for the probability of survival can now be formulated by (4), (7) and

$$\phi(u, b_{max}) = \frac{\phi(u)}{\phi(b_{max})}, \quad (12)$$

where  $0 \leq u \leq b \leq b_{max}$  and as before  $\phi(u)$  is the probability of survival in absence of the barrier.

**Example:** For exponentially distributed claim amounts ( $F(z) = 1 - e^{-z}$ ), equation (4) can be expressed as a hyperbolic partial differential equation with variable coefficients

$$c \frac{\partial^2 \phi}{\partial u^2} + \frac{1}{\alpha m b^{m-1}} \frac{\partial^2 \phi}{\partial b \partial u} + (c - \lambda) \frac{\partial \phi}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial \phi}{\partial b} = 0 \quad (13)$$

and with boundary conditions (7) and

$$\left( c \frac{\partial \phi}{\partial u} + \frac{1}{\alpha m b^{m-1}} \frac{\partial \phi}{\partial b} - \lambda \phi \right) \Big|_{u=0} = 0. \quad (14)$$

Since  $\phi_0(u, b) = e^{-sb^m - r(s)u}$  is a solution of (13), where  $r(s)$  satisfies

$$cr^2 + \left( s/\alpha + \lambda - c \right) r - s/\alpha = 0, \quad (15)$$

one can try to obtain a solution of the form

$$\phi(u, b) = \int_0^\infty e^{-sb^m} A_1(s) e^{-r_1(s)u} ds + \int_0^\infty e^{-sb^m} A_2(s) e^{-r_2(s)u} ds + \phi(u),$$

where  $r_1(s), r_2(s)$  are the solutions of (15) and the  $A_i(s)$  have to be determined according to (7) and (14). However, this turns out to be an intricate problem.

Similarly, the integro-differential equations for  $W(u, b)$  and  $V(u, b)$  can be expressed as second-order PDE's in the case of exponentially distributed claims.

### 3 Solution techniques

The above example shows that even for the simple case of exponentially distributed claim amounts it is a delicate problem to obtain analytical solutions. Thus there is a need for effective algorithms to obtain numerical solutions to these problems. In this paper we focus on the development of number-theoretic solution methods.

Following a procedure developed by GERBER [14] for the case of linear barriers, we first show that the boundary value problem (9) together with (10) has a unique bounded solution. For that purpose, we define an operator  $A$  by

$$\begin{aligned} Ag(u, b) = & \int_0^{t^*} \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct} g \left( u + ct - z, \left( b^m + \frac{t}{\alpha} \right)^{1/m} \right) dF(z) dt + \\ & + \int_{t^*}^\infty \lambda e^{-(\lambda+\delta)t} \int_0^{\left( b^m + \frac{t}{\alpha} \right)^{1/m}} g \left( \left( b^m + \frac{t}{\alpha} \right)^{1/m} - z, \left( b^m + \frac{t}{\alpha} \right)^{1/m} \right) dF(z) dt + \\ & + \int_{t^*}^\infty \lambda e^{-\lambda t} \int_{t^*}^t e^{-\delta s} \left( c - \frac{1}{m\alpha \left( b^m + \frac{s}{\alpha} \right)^{1-1/m}} \right) ds dt. \quad (16) \end{aligned}$$

Here  $t^*$  is the positive solution of  $u + ct = \left( b^m + \frac{t}{\alpha} \right)^{1/m}$  (since  $m > 1$ ,  $b_t$  is concave and  $u \leq b$ , so this number is unique). Note that (16) can be interpreted as a conditioning on

whether a claim occurs before the surplus process hits the dividend barrier ( $t < t^*$ ) or after this event (in which case we have an additional term representing the discounted dividends paid until the claim occurs). The solution  $W(u, b)$  of (9) with its initial condition (10) is a fixed point of the integral operator  $A$ . For any two bounded functions  $g_1, g_2$

$$|Ag_1(u, b) - Ag_2(u, b)| \leq \|g_1 - g_2\|_\infty \int_0^\infty \lambda e^{-(\lambda+\delta)t} dt \leq \frac{\lambda}{\lambda+\delta} \|g_1 - g_2\|_\infty \quad (17)$$

for arbitrary  $0 \leq u \leq b < \infty$ , where  $\|\cdot\|_\infty$  is the supremum norm on  $0 \leq u \leq b < \infty$ , and thus it follows that  $A$  is a contraction and the fixed point is unique by Banach's theorem.

Proceeding in the same way as for  $W(u, b)$  above, one can easily show that equation (11) together with its initial condition has a unique bounded solution.

In the case of Model B we can proceed in a similar way to obtain a contraction map for the probability of survival as its fixed point: Like in equation (16), let  $t^*$  be the time when the surplus would reach the dividend barrier given that no claim occurs. Let furthermore  $t^{**} = \alpha(b_{max}^m - b^m)$  be the time when the dividend barrier reaches the absorbing barrier, and  $\tilde{t} = (b_{max} - u)/c$  the time when the surplus would reach the absorbing barrier in the absence of a dividend barrier and of claims. As the dividend barrier is an increasing function on  $\mathbb{R}^+$ ,  $t^{**}$  is uniquely determined, just as is  $\tilde{t}$ . Combining the two possible scenarios  $0 \leq t^{**} \leq \tilde{t} \leq t^*$  and  $0 \leq t^* \leq \tilde{t} \leq t^{**}$  (depending on the values of  $u$  and  $b$ ), we define the operator  $A$  as

$$A\phi(u, b) = \int_0^T \lambda e^{-\lambda t} \int_0^{z_{min}(u, b, t)} \phi \left( z_{min}(u, b, t) - z, \left( b^m + \frac{t}{\alpha} \right)^{\frac{1}{m}} \right) dF(z) dt + e^{-\lambda T}, \quad (18)$$

where  $T = \max(\tilde{t}, t^{**})$  is the time when the surplus process would reach the absorbing upper barrier  $b_{max}$ , and

$$z_{min}(u, b, t) = \min \left( u + ct, \left( b^m + \frac{t}{\alpha} \right)^{\frac{1}{m}} \right). \quad (19)$$

Let  $\phi_1$  and  $\phi_2$  now be two bounded functions on  $0 \leq u \leq b \leq b_{max}$ , then

$$|A\phi_1(u, b) - A\phi_2(u, b)| \leq \|\phi_1 - \phi_2\|_\infty \int_0^T \lambda e^{-\lambda t} dt = \|\phi_1 - \phi_2\|_\infty (1 - e^{-\lambda T}).$$

Since  $T = T(u, b) < M < \infty$ , this operator is a contraction, and Banach's fixed point theorem establishes the existence and uniqueness of the solution. Here, the absorbing barrier and the resulting restriction to the finite area  $0 \leq u \leq b \leq b_{max}$  ensures that the solution is unique in contrast to the case without the absorbing barrier.

Correspondingly, the contraction map for the expected sum of dividend payments in Model

B is given by

$$\begin{aligned}
Ag(u, b) = & \int_0^{t^*} \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct} g\left(u+ct-z, \left(b^m + \frac{t}{\alpha}\right)^{1/m}\right) dF(z) dt + \\
& + \int_{t^*}^{t^{**}} \lambda e^{-(\lambda+\delta)t} \int_0^{\left(b^m + \frac{t}{\alpha}\right)^{1/m}} g\left(\left(b^m + \frac{t}{\alpha}\right)^{1/m} - z, \left(b^m + \frac{t}{\alpha}\right)^{1/m}\right) dF(z) dt + \\
& + \int_{t^*}^{t^{**}} e^{-(\lambda+\delta)t} \left(c - \frac{1}{m\alpha \left(b^m + \frac{t}{\alpha}\right)^{1-1/m}}\right) dt, \quad (20)
\end{aligned}$$

if  $t^{**} > t^*$  and  $Ag(u, b) = 0$  otherwise, because then the surplus reaches the absorbing barrier before the dividend barrier. The last term in (20) represents the dividends that are paid out until  $t^{**}$  and is a simplification of the original expression

$$\begin{aligned}
& \int_{t^*}^{t^{**}} \lambda e^{-\lambda t} \int_{t^*}^t e^{-\delta s} \left(c - \frac{1}{m\alpha \left(b^m + \frac{s}{\alpha}\right)^{1-1/m}}\right) ds dt + \\
& \int_{t^{**}}^{\infty} \lambda e^{-\lambda t} \int_{t^*}^{t^{**}} e^{-\delta s} \left(c - \frac{1}{m\alpha \left(b^m + \frac{s}{\alpha}\right)^{1-1/m}}\right) ds dt.
\end{aligned}$$

From (20) it follows that

$$\|Ag_1(u, b) - Ag_2(u, b)\|_{\infty} \leq \frac{\lambda}{\lambda + \delta} \left(1 - e^{-(\lambda+\delta)t^{**}}\right) \|g_1 - g_2\|_{\infty},$$

for any two bounded functions  $g_1, g_2$  and we again have a contraction in the Banach space of bounded functions equipped with the supremum norm, which implies the existence and uniqueness of the solution.

The following algorithms are now efficient ways of approximating the corresponding fixed point:

### 3.1 Double-recursive Algorithm

This procedure will be described for the operator (16); it can easily be adapted to the other integral operators introduced above. Moreover we will restrict ourselves to the case of exponentially distributed claim amounts (with parameter  $\gamma$ ); the extension of the method to other distributions is straightforward.

The fixed point of (16) can be approximated by applying the contracting integral operator  $A$   $k$  times to a starting function  $h(u, b)$  which we choose to be the inhomogeneous term in the corresponding integral operator (where  $k$  is chosen according to the desired accuracy of the solution):

$$\begin{aligned}
g^{(k)}(u, b) &= A^k g^{(0)}(u, b), \\
g^{(0)}(u, b) &= h(u, b) := \int_{t^*}^{\infty} \lambda e^{-\lambda t} \int_{t^*}^t e^{-\delta s} \left(c - \frac{1}{m\alpha \left(b^m + \frac{s}{\alpha}\right)^{1-\frac{1}{m}}}\right) ds dt.
\end{aligned}$$

This leads to a  $2k$ -dimensional integral for  $g^{(k)}(u, b)$ , which is calculated numerically using Monte Carlo and Quasi-Monte Carlo methods. For that purpose we transform the integration domain of operator (16) into the unit cube:

$$\begin{aligned}
Ag(u, b) &= h(u, b) + \\
&+ \frac{\lambda}{\lambda + \delta} \left[ \left(1 - e^{-(\lambda + \delta)t^*}\right) \int_0^1 \int_0^1 g \left( u + ct_1 - z_1, \left( b^m + \frac{t_1}{\alpha} \right)^{\frac{1}{m}} \right) \left(1 - e^{-\gamma(u + ct_1)}\right) dv_1 dw_1 \right. \\
&\left. + e^{-(\lambda + \delta)t^*} \int_0^1 \int_0^1 g \left( \left( b^m + \frac{t_2}{\alpha} \right)^{\frac{1}{m}} - z_2, \left( b^m + \frac{t_2}{\alpha} \right)^{\frac{1}{m}} \right) \cdot \left(1 - e^{-\gamma \left( b^m + \frac{t_2}{\alpha} \right)^{\frac{1}{m}}}\right) dv_2 dw_2 \right]
\end{aligned}$$

with

$$t_1 = -\frac{\log(1 - w_1 (1 - e^{-(\lambda + \delta)t^*}))}{\lambda + \delta} \quad z_1 = -\frac{\log(1 - v_1 (1 - e^{-\gamma(u + ct_1)}))}{\gamma} \quad (21)$$

$$t_2 = t^* - \frac{\log(1 - w_2)}{(\lambda + \delta)} \quad z_2 = -\frac{\log\left(1 - v_2 \left(1 - e^{-\gamma \left( b^m + \frac{t_2}{\alpha} \right)^{\frac{1}{m}}}\right)\right)}{\gamma}. \quad (22)$$

The Monte Carlo-estimator of  $W(u, b)$  for given values of  $u$  and  $b$  is

$$W(u, b) \approx \frac{1}{N} \sum_{n=1}^N g_n^{(k)}(u, b), \quad (23)$$

where the  $g_n^{(k)}(u, b)$  are calculated recursively for each  $n$  by

$$g_n^{(0)}(u, b) = h(u, b)$$

and

$$\begin{aligned}
g_n^{(i)}(u, b) &= h(u, b) + \frac{\lambda}{\lambda + \delta} \cdot \\
&\cdot \left\{ \left(1 - e^{-\gamma(u + ct_{1,n}^i)}\right) \left(1 - e^{-(\lambda + \delta)t^*}\right) g_n^{(i-1)} \left( u + ct_{1,n}^i - z_{1,n}^i, \left( b^m + \frac{t_{1,n}^i}{\alpha} \right)^{\frac{1}{m}} \right) + \right. \\
&\left. + \left(1 - e^{-\gamma \left( b^m + \frac{t_{2,n}^i}{\alpha} \right)^{\frac{1}{m}}}\right) e^{-(\lambda + \delta)t^*} g_n^{(i-1)} \left( \left( b^m + \frac{t_{2,n}^i}{\alpha} \right)^{\frac{1}{m}} - z_{2,n}^i, \left( b^m + \frac{t_{2,n}^i}{\alpha} \right)^{\frac{1}{m}} \right) \right\}.
\end{aligned}$$

Here  $t_{j,n}^i$  and  $z_{j,n}^i$  ( $j = 1, 2$ ) are determined according to (21) and (22) for (quasi-)random deviates  $v_j, w_j$  of the uniform distribution in the unit interval ( $1 \leq i \leq k$ ).

Since in every recursion step the function  $g$  is called twice, the number of evaluations of  $g$  doubles in every recursion step. Thus, in order to keep the computations tractable, in what we will call the double-recursive algorithm in the sequel, the double recursion is only used for the first two recursive steps and for the remaining recursion steps the recursive algorithm described in Section 3.2 is applied.



### 3.2 Recursive Algorithm

Instead of calculating the first two integrals occurring in operator (16) separately, one can combine them to one integral. A suitable change of variables then leads to

$$Ag(u, b) = h(u, b) + \int_0^1 \int_0^1 \frac{\lambda}{\lambda + \delta} \left(1 - e^{-\gamma z_{\min}(u, b, t)}\right) g \left( z_{\min}(u, b, t) - z, \left(b^m + \frac{t}{\alpha}\right)^{\frac{1}{m}} \right) dv dw \quad (24)$$

where  $t$  and  $z$  are given by

$$\begin{aligned} t &= -\frac{\log(1-w)}{(\lambda + \delta)} \\ z &= -\frac{\log(1-v(1 - e^{-\gamma z_{\min}(u, b, t)}))}{\gamma} \end{aligned} \quad (25)$$

and  $z_{\min}(u, b, t)$  is determined by (19). Like in the double recursive case, this integral operator is now applied  $k$  times onto  $g^{(0)}$ , and the resulting multidimensional integral  $g^{(k)}(u, b)$  is again approximated by

$$g^{(k)}(u, b) \approx \frac{1}{N} \sum_{n=1}^N g_n^{(k)}(u, b), \quad (26)$$

where each  $g_n^{(k)}(u, b)$  ( $n = 1, \dots, N$ ) is based on a pseudo-random (or quasi-random, resp.) point  $\mathbf{x}_n \in [0, 1]^{2k}$  and calculated by the recursion

$$\begin{aligned} g_k^{(0)}(u, b) &= h(u, b), \\ g_n^{(i)}(u, b) &= \frac{\lambda}{\lambda + \delta} \left(1 - e^{-\gamma z_{\min}(u, b, t_n^i)}\right) g_n^{i-1} \left( z_{\min}(u, b, t_n^i) - z_n^i, \left(b^m + \frac{t_n^i}{\alpha}\right)^{\frac{1}{m}} \right) + h(u, b), \end{aligned}$$

with  $1 \leq i \leq k$ .  $t_n^i$  and  $z_n^i$  are given by (25) with  $v$  and  $w$  being the value of the  $2i$ -th and  $2i + 1$ -th, component of  $\mathbf{x}_n$ , respectively. Note that for this algorithm, the number of integration points needed for a given recursion depth is one fourth of the corresponding number required for the double-recursive case.

### 3.3 Simulation

Since there are no analytical solutions available for the above problems, we need simulation estimates of the ruin probabilities and discounted dividend payments to compare them to the results of the integration methods that were described in the last sections.

We sample  $N$  paths of the risk reserve process in the following way: Starting with  $t_0 := 0$ ,  $b_0 := b$  and  $x_0 := u$ , where  $u$  is the initial reserve of the insurance company, we first generate an exponentially distributed random variable  $\tilde{t}_i$  with parameter  $\lambda$  for the time until the next claim occurs and set  $t_{i+1} := t_i + \tilde{t}_i$ . The claim amount is sampled from an exponentially distributed random variable  $z_i$  (with parameter  $\gamma$ ), and the reserve after the claim is  $x_{i+1} := \min\{x_i + c\tilde{t}_i, (b_i^m + \tilde{t}_i/\alpha)^{1/m}\} - z_i$ . Due to the structure of the dividend

barrier, we can reset the origin to  $t_{i+1}$  in every step, if we also set  $b_{i+1} = \left(b_i^m + \frac{\tilde{t}_i}{\alpha}\right)^{1/m}$ . We then have to discount the dividend payments between the  $i$ -th and  $(i+1)$ -th claims by the factor  $e^{-\delta t_i}$ .

A simulation estimate for the survival probability  $\phi(u, b)$  can now be obtained by

$$\phi(u, b) \approx \frac{m}{N},$$

where  $m$  is the number of paths for which ruin does not occur (i.e.  $x_i > 0 \forall i$ ). We consider a path as having survived, if for some  $i$  the condition  $x_i > x_{max}$  is fulfilled, where  $x_{max}$  is a sufficiently large threshold. This can be viewed as an absorbing horizontal barrier at  $x_{max}$ , and so the process stops with probability 1. Using this stopping criterion, we overestimate the actual probability of survival  $\phi(u, b)$ ; for sufficiently large  $x_{max}$ , however, this effect is negligible.

For the simulation of the expected value of the dividend payments, we proceed as described above and whenever the process reaches the dividend barrier, i.e.  $x_i + c\tilde{t}_i > (b_i^m + \tilde{t}_i/\alpha)^{\frac{1}{m}}$ , we need to calculate the amount of dividends that are paid until the next claim  $i$  occurs:

$$v_i := v_{i-1} + e^{-\delta t_i} \int_{t^*}^{\tilde{t}_i} e^{-\delta s} \left( c - \frac{1}{m\alpha (b_i^m + \frac{s}{\alpha})^{1-\frac{1}{m}}} \right) ds, \quad i \geq 1$$

and  $v_0 = 0$ , where  $t^*$  is the positive solution of  $x_i + ct = (b_i^m + \frac{t}{\alpha})^{1/m}$ , i.e. the time when the process reaches the dividend barrier. The process is stopped, if ruin occurs (i.e.  $x_i < 0$  for some  $i$ ) or at some sufficiently large time  $t_{max}$ , after which the expected value of discounted dividends becomes negligible due to the discount factor  $e^{-\delta t}$ . Let  $v(j)$  now be the final value of  $v_i$  for path  $j$ . The expected value of the dividends is then approximated by

$$E[W(u, b)] \approx \frac{1}{N} \sum_{j=1}^N v(j).$$

### 3.4 Quasi-Monte Carlo Approach

The use of deterministic uniformly distributed point sequences (instead of pseudo-random sequences in crude Monte Carlo) has proven to be an efficient extension of the classical Monte Carlo method. A well-known measure for the uniformness of the distribution of a sequence  $\{\mathbf{x}_n\}_{1 \leq n \leq N}$  in  $U^s := [0, 1]^s$  is the star-discrepancy

$$D_N^*(\mathbf{x}_n) = \sup_{I \in J_0^s} \left| \frac{A(\mathbf{x}_n; I)}{N} - \lambda_s(I) \right|,$$

where  $J_0^s$  is the set of all intervals of the form  $[0, \vec{y}) = [0, y_1) \times [0, y_2) \times \dots \times [0, y_s)$  with  $0 \leq y_i < 1$ ,  $i = 1, \dots, s$  and  $A(\mathbf{x}_n; I)$  is the number of points of the sequence  $\{\mathbf{x}_n\}_{1 \leq n \leq N}$  that lie in  $I$ .  $\lambda_s(I)$  denotes the  $s$ -dimensional Lebesgue-measure of  $I$ .

The notion of discrepancy is particularly useful for obtaining an upper bound for the error of quasi-Monte Carlo integration:

**Lemma 1 (Koksma-Hlawka Inequality).** *Let the function  $f : [0, 1]^s \rightarrow \mathbb{R}$  be of bounded variation  $V([0, 1]^s, f)$  in the sense of Hardy and Krause. Then for any set of points  $\{x_1, \dots, x_N\} \subset [0, 1]^s$*

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_{[0,1]^s} f(u) du \right| \leq V([0, 1]^s, f) D_N^*(x_1, \dots, x_N) . \quad (27)$$

For a proof of this famous inequality we refer to [10]. This error bound is deterministic (opposed to error bounds obtainable for crude Monte Carlo). Especially for  $s$  not too large, certain Quasi-Monte Carlo sequences have turned out to be superior to pseudo-Monte Carlo sequences in many applications. This is in particular the case for so-called *low discrepancy sequences*, i.e. sequences for which

$$D_N^*(x_1, \dots, x_N) \leq C_s \frac{(\log N)^s}{N}, \quad (28)$$

with an explicitly computable constant  $C_s$ , holds. Bounds for  $C_s$  are usually pessimistic and often the actual error made by Quasi-Monte Carlo integration is much lower than the bound implied by  $C_s$  (see e.g. [8]). Some low discrepancy sequences will be given in the sequel:

- The Halton sequence [16] is defined as a sequence of vectors in  $U^s$  based on the digit representation of  $n$  in base  $p_i$

$$\xi_n = (b_{p_1}(n), b_{p_2}(n), \dots, b_{p_s}(n)), \quad (29)$$

where  $p_i$  is the  $i$ th prime number and  $b_p(n)$  is the digit reversal function for base  $p$  given by

$$b_p(n) = \sum_{k=0}^{\infty} n_k p^{-k-1}, \quad n = \sum_{k=0}^{\infty} n_k p^k,$$

where the  $n_k$  are integers. One could also use pairwise coprime base numbers, but the error estimate turns out to be the best possible for prime bases  $p_n$ .

Better error bounds can be obtained for low-discrepancy sequences based on so-called  $(t, m, s)$ -nets or nets for short. These nets are based on the  $b$ -adic representation of vectors in  $U^s$ . Instead of optimizing the discrepancy itself, one considers the discrepancy with respect to elementary intervals  $J$  in base  $b$  only, i.e.  $J = \prod_{i=1}^s [a_i b^{-d_i}, (a_i + 1) b^{-d_i}]$  with integers  $d_i \geq 0$  and integers  $0 \leq a_i < b^{d_i}$  for  $1 \leq i \leq s$ , and tries to construct point sequences in  $U^s$  such that the discrepancy with respect to these intervals  $J$  is optimal for subsequences of length  $N = b^m$ .

Let  $\#(J, N)$  denote the number of points of a sequence  $\{\mathbf{x}_n\}_{1 \leq n \leq N}$  that lie in  $J$ . A point set  $\mathcal{P}$  with  $\text{card}(\mathcal{P}) = b^m$  is now called a  $(t, m, s)$ -net, if

$$\#(J, b^m) = b^t$$

for every elementary interval  $J$  with  $\lambda_s(J) = b^{t-m}$ . The parameter  $t$  is a quality parameter. For  $t = 0$  we have the minimal discrepancy of the point set  $\mathcal{P}$  with respect to the

family of elementary intervals.

**Definition:** Let  $t \geq 0$  be an integer. A sequence  $\xi_1, \xi_2, \dots$  of points in  $U^s$  is called a  $(t, s)$ -sequence in base  $b$ , if for all integers  $k \geq 0$  and  $m > t$ , the point set consisting of the  $\xi_n$  with  $kb^m < n \leq (k+1)b^m$  is a  $(t, m, s)$ -net in base  $b$ .

Examples of  $(t, m, s)$ -nets are:

- The Sobol Sequence is a  $(t, s)$ -sequence in base 2 with values  $t$  that depend on  $s$ . For a construction of this sequence we refer to [23].
- The Niederreiter sequences (cf. [18]) yield  $(t, s)$ -sequences in arbitrary base; among them there are  $(0, s)$ -sequences in prime power bases  $b \geq s$ . In particular, for Niederreiter sequences the constant  $C_s$  in (28) tends to zero for  $s \rightarrow \infty$ .

Following a technique developed in [25], we can now use (27) to find an upper bound for the error of the recursive algorithm estimate introduced in Section 3.2 in terms of the discrepancy of the sequence used:

**Theorem 1.** *If the expected value  $W(u, b)$  of the discounted dividends is approximated by  $g^{(k)}(u, b)$  as given in (23) using a sequence  $\omega$  of  $N$  elements, the error is bounded by*

$$\left\| W(u, b) - g^{(k)}(u, b) \right\|_{\infty} \leq \frac{\|h(u, b)\|_{\infty}}{1 - q} \left( q^k + qD_N(\omega) \right) \quad (30)$$

with  $q := \frac{\lambda}{\lambda + \delta}$ .

*Proof.* Since we have  $g^{(0)}(u, b) = h(u, b)$ , it follows from Banach's fixed point theorem together with the estimate (17), that

$$\begin{aligned} \left\| W(u, b) - g^{(k)}(u, b) \right\|_{\infty} &\leq \left\| W(u, b) - A^k h(u, b) \right\|_{\infty} + \left\| A^k h(u, b) - g^{(k)}(u, b) \right\|_{\infty} \\ &\leq \frac{q^k}{1 - q} \|h(u, b)\|_{\infty} + \left\| A^k h(u, b) - g^{(k)}(u, b) \right\|_{\infty} \end{aligned} \quad (31)$$

Iterating the integral equation (24)  $k$  times leads to

$$A^k h(u_0, b_0) = \sum_{i=1}^k \int_{[0,1]^{2i}} \cdots \int \left( \prod_{j=0}^{i-1} C_j q \right) h(u_i, b_i) dv_{i-1} dw_{i-1} \dots dv_0 dw_0 + h(u_0, b_0) \quad (32a)$$

$$= \int_{[0,1]^{2k}} \cdots \int \sum_{i=1}^k \left( \prod_{j=0}^{i-1} C_j q \right) h(u_i, b_i) dv_{k-1} dw_{k-1} \dots dv_0 dw_0 + h(u_0, b_0) \quad (32b)$$

where for  $0 \leq j \leq k-1$  we have

$$\begin{aligned}
t_j &= -\frac{1}{\lambda} \log(1 - w_j), \\
z_j &= -\frac{1}{\gamma} \log \left( 1 - v_j \left( 1 - e^{-\gamma z_{\min}(u_j, b_j, t_j)} \right) \right), \\
z_{\min}(u_j, b_j, t_j) &= \text{cut}_j := \min \left( u_j + ct_j, \left( b_j^m + \frac{t_j}{\alpha} \right)^{\frac{1}{m}} \right), \\
C_j &= (1 - e^{-\gamma \text{cut}_j}), \\
u_{j+1} &= \text{cut}_j - z_j, \\
b_{j+1} &= \left( b_j^m + \frac{t_j}{\alpha} \right)^{\frac{1}{m}}.
\end{aligned} \tag{33}$$

In our recursive algorithm the  $2k$ -dimensional integral (32b) is approximated by quasi-Monte Carlo integration and in order to use Koksma-Hlawka's inequality for bounding the error, we have to determine the total variation of the integrand in (32b). For that purpose we investigate each of the summands separately and define  $F_i$  to be the integrand of the  $i$ -th term in (32a):

$$F_i(v_0, w_0, \dots, v_{i-1}, w_{i-1}) := q^i \prod_{j=0}^{i-1} \left( 1 - e^{-\gamma z_{\min}(u_j, b_j, t_j)} \right) h(u_i, b_i). \tag{34}$$

We now show that this function is increasing in all the variables  $w_j$  and decreasing in all the variables  $v_j$  ( $j = 1, \dots, i-1$ ):

Choose a  $j \in \{1, \dots, i-1\}$  and let  $v_j$  be increasing (while all the other variables are fixed), then  $C_k$ ,  $u_k$ , and  $t_k$  remain constant for all  $k \leq j$ . Furthermore  $z_k$  remains constant for all  $k < j$  and so does  $b_k$  for arbitrary  $k$ . But then it is easy to see that  $u_{j+1}$  and  $z_{\min}(u_{j+1}, b_{j+1}, t_{j+1})$  are decreasing. By induction and some elementary monotonicity investigations it follows that  $u_{j+k}$  and  $z_{\min}(u_{j+k}, b_{j+k}, t_{j+k})$  are decreasing for all  $k \geq 1$ . But since  $h(u, b)$  is an increasing function of  $u$  it follows from (34) that  $F_i$  is a decreasing function of  $v_j$  ( $j = 1, \dots, i-1$ ). Similarly it can be shown that  $F_i$  is an increasing function of  $w_j$  ( $j = 1, \dots, i-1$ ).

This monotone behavior now allows to bound the variation of  $F_i$ :

$$V([0, 1]^{2i}; F_i) = F_i(0, 1, \dots, 0, 1) - F_i(1, 0, \dots, 1, 0) \leq \left( \frac{\lambda}{\lambda + \delta} \right)^i \|h\|_{\infty}$$

By summing up the variations of the  $F_i$  we get an upper bound for the total variation of the integrand  $F$  of (32b)

$$V([0, 1]^{2k}; F) \leq \sum_{i=1}^k \left( \frac{\lambda}{\lambda + \delta} \right)^i \|h\|_{\infty} = q \frac{1 - q^k}{1 - q} \|h\|_{\infty} \leq \frac{q}{1 - q} \|h\|_{\infty}.$$

If we use this estimate together with Lemma 1 we get

$$\left\| A^k h(u, b) - g^{(k)}(u, b) \right\|_{\infty} \leq \|h\|_{\infty} \frac{q}{1 - q} D_N(\omega)$$

and inserting this into equation (31) finally gives

$$\left\| W(u, b) - g^{(k)}(u, b) \right\|_{\infty} \leq \frac{q^k}{1-q} \|h(u, b)\|_{\infty} + \|h\|_{\infty} \frac{q}{1-q} D_N(\omega) = \frac{\|h\|_{\infty}}{1-q} \left( q^k + q D_N(\omega) \right).$$

□

## 4 Numerical results for the parabolic case

In this section numerical illustrations for a parabolic dividend barrier of the form  $b_t = \sqrt{b^2 + t/\alpha}$  and exponentially distributed claim amounts ( $F(z) = 1 - e^{-z}$ ) are presented. Note that in this case

$$t^* = \frac{1}{2\alpha c^2} - \frac{u}{c} + \sqrt{\left(\frac{1}{2\alpha c^2} - \frac{u}{c}\right)^2 + \frac{b^2 - u^2}{c^2}}$$

and the inhomogeneous term  $h(u, b)$  in (16) can be calculated explicitly to

$$h(u, b) = e^{-t^*(\lambda+\delta)} \left( \frac{c}{\lambda+\delta} - \sqrt{\frac{\pi}{(\lambda+\delta)\alpha}} \frac{e^{z^2}}{2} \operatorname{erfc}(z) \right)$$

with  $z = \sqrt{(\lambda+\delta)(\alpha b^2 + t^*)}$  and thus we have  $\|h(u, b)\|_{\infty} \leq \frac{c}{\lambda+\delta}$ .

The parameters are set to  $c = 1.5$ ,  $\delta = 0.1$ ,  $\alpha = 0.5$ ,  $\lambda = \gamma = 1$  and the absorbing upper barrier in Model B is chosen at  $b_{\max} = 4$ .

The MC and QMC estimators are obtained using  $N = 66\,000$  paths for the recursive case and for the simulation and  $N = 33\,000$  for the double-recursive calculations. The corresponding "exact" value, in lack of an analytic solution, is obtained by a MC-simulation over 10 million paths for each choice of  $u$  and  $b$ .

For the recursive and double recursive calculations we use a recursion depth of  $k = 66$ , which leads to a 132-dimensional sequence needed for the MC- and QMC-calculations, while for the simulation we take a 400-dimensional sequence so that 200 consecutive claims and interoccurrence times of a risk reserve sample path can use the different dimensions of one element of the sequence and correlations among the claim sizes and claim occurrence times are avoided.

We use so-called hybrid Monte Carlo sequences for all our QMC-calculations, where the initial 50 dimensions are generated by a 50-dimensional low discrepancy sequence and the remaining dimensions are generated by a pseudo-random number generator. Throughout this paper, we use `ran2` as our pseudo-random number generator, which basically is an improved version of a Minimal Standard generator based on a multiplicative congruential algorithm (for a description we refer to [20]). The use of hybrid Monte Carlo sequences has proven to be a successful modification of the QMC-technique, since for low discrepancy sequences typically the number of points needed to obtain a satisfying degree of uniformness dramatically increases with the number of dimensions.

The different methods and sequences used are compared via the mean square error

$$S = \sqrt{\frac{1}{|P|} \sum_{(u,b) \in P} (g(u,b) - \tilde{g}(u,b))^2},$$

where  $g(u,b)$  and  $\tilde{g}(u,b)$  denote the exact and the approximated value, respectively, and the set  $P$  is a grid in the triangular region ( $b = 0..[0.1]..1, u = 0..[0.1]..b$ ). In addition, for each method we give the maximal deviation of the approximated value from the corresponding exact value  $\|\Delta\|_\infty = \max_{(u,b) \in P} (g(u,b) - \tilde{g}(u,b))$ .

#### 4.1 Survival probability

In Model A the survival probability can only be calculated using the simulation approach. Table 1 gives the mean-square and the maximal error of the simulation results (together with the corresponding calculation time in seconds) for each of the sequences used (with  $N = 66\,000$ ):

		Monte Carlo	Halton	Niederr. (t,s)	Sobol
Simulation	$S$	0.001307	0.001798	0.001706	0.0009
	$\ \Delta\ _\infty$	0.003741	0.003619	0.003472	0.002451
		(163.16 s)	(149.58 s)	(281.61 s)	(150.09 s)

TABLE 1: Simulation errors for the survival probability in Model A

Figure 2 shows a log-log-plot of the mean square error  $S$  as a function of  $N$ . To quantify the effect of using a low discrepancy sequence, we perform a regression analysis by fitting

$$\log_2(S) = a_0 + a_1 \log_2(N) + a_2 \log_2(\log_2(N)) + \epsilon$$

to the data using a least square fit. Note that Koksma-Hlawka's inequality (27) could be interpreted as implying  $a_1 = -1$  and  $a_2 = s$ , where  $s$  is the dimension of the sequence used. However, since we use a hybrid sequence and since the effective dimension may differ from the theoretical dimension, the values of  $a_1$  and  $a_2$  deviate from the ones above. Figure 3 gives these fitted curves. In the sequel all figures on simulation results will be given in terms of their regression fits.

In Model B approximate solutions for the survival probability can be obtained by the recursive method using the operator (18) and by simulation. The numerical errors and the corresponding calculation time are given in Table 3 and the fitted curves for the mean square error are depicted in Figure 4.

Figure 4 shows that while the recursive Monte Carlo method is favorable to the Monte Carlo simulation, for larger values of  $N$  the simulation technique using the Halton and the Sobol sequence, respectively, gives even better results. However, the best results in terms of convergence rate of the error are obtained for the recursive method using Quasi-Monte

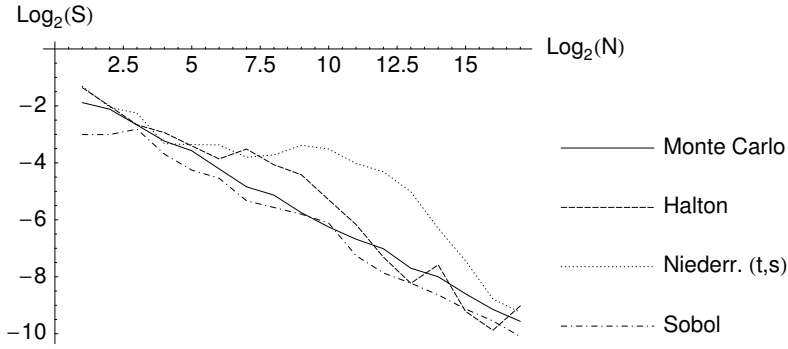


FIGURE 2: Mean square error of the simulated survival probability in Model A

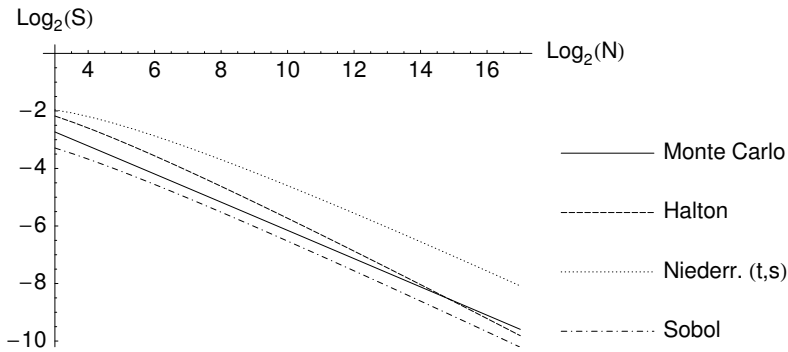


FIGURE 3: Fits of the simulated survival probability in Model A

$b \setminus x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	10.28										
0.1	10.28	10.75									
0.2	10.32	10.77	11.19								
0.3	10.40	10.86	11.27	11.63							
0.4	10.51	10.99	11.42	11.75	12.04						
0.5	10.62	11.12	11.56	11.94	12.24	12.44					
0.6	10.78	11.28	11.74	12.16	12.48	12.75	12.90				
0.7	10.94	11.48	11.97	12.37	12.74	13.01	13.25	13.35			
0.8	11.13	11.69	12.19	12.65	13.02	13.34	13.60	13.79	13.86		
0.9	11.33	11.91	12.44	12.91	13.31	13.67	13.95	14.22	14.37	14.43	
1.0	11.54	12.14	12.68	13.18	13.60	14.02	14.35	14.62	14.83	14.97	15.01

TABLE 2: Exact values of the survival probability in % in Model A



		Monte Carlo	Halton	Niederr. (t,s)	Sobol
Simulation	$S$	0.001796	0.000676	0.001621	0.00062
	$\ \Delta\ _\infty$	0.004066 (99.71 s)	0.001813 (86.92 s)	0.002529 (87.21 s)	0.001217 (86.91 s)
Recursive	$S$	0.000934	0.000155	0.000168	0.000128
	$\ \Delta\ _\infty$	0.002504 (386.44 s)	0.000365 (374.3 s)	0.000392 (374.4 s)	0.000317 (374.21 s)

TABLE 3: Errors for the survival probability in Model B

Carlo sequences. To quantify this effect, we introduce the efficiency gain

$$gain_i = \frac{N_{MC}^*(S)}{N_i^*(S)}$$

where  $N_{MC}^*(S)$  is the number of paths needed in the Monte Carlo simulation to reach a given error of  $S$ , and  $N_i^*(S)$  is the corresponding number of paths (the number of summands in approximations (23) and (26), respectively) using an alternative method. Figure 5 shows that except for the  $(0, s)$ -nets all methods are an improvement in efficiency compared to Monte Carlo simulation, and the gain increases with smaller errors.

## 4.2 Expected value of the dividend payments

The exact values of  $W(u, b)$  in Models A and B are given in Tables 5 and 6, respectively. The numerical results given in Table 7 and Figures 6 and 7 show that the performance of the various solution techniques is similar to the case of survival probabilities. For a moderate choice of  $N$  ( $N \leq 2^{10}$ ) the Monte Carlo methods have a smaller mean square error than the QMC simulation techniques; for larger  $N$ , however, all Quasi-Monte Carlo methods outperform the Monte Carlo schemes, with the recursive algorithm giving better results than the simulation. This is in particular relevant for practical purposes, since the generation of these QMC-sequences can be done faster than the generation of pseudo-random numbers based on `ran1` or `ran2`.

For the dividend payments in Model B the superiority of the Quasi-Monte Carlo approach is even more pronounced (see Figures 8,9 and Table 8).

Since for a fixed  $N$  the recursive numerical techniques need more calculation time than the simulation approach, it is instructive to investigate the accuracy of the numerical results with respect to calculation time. Figure 10 gives a log-log-plot of the mean-square error  $S$  as a function of calculation time  $t$  for the dividend payments in Model B. It turns out that the Quasi-Monte Carlo techniques clearly outperform the corresponding Monte Carlo techniques. For smaller values of  $t$  the Sobol sequence is particularly well-suited for our integrands, whereas for large  $t$  the use of the Halton sequence seems preferable.

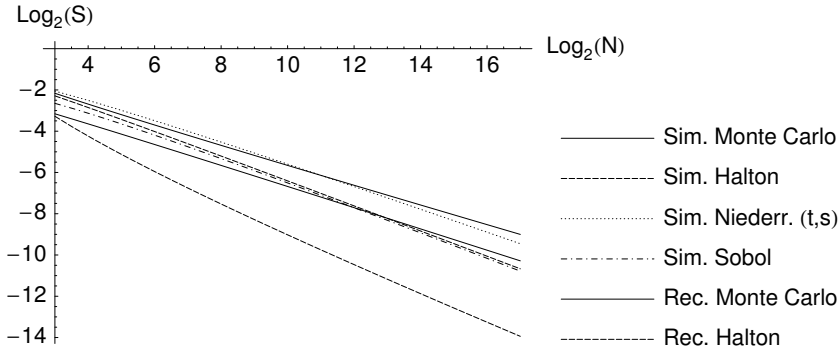


FIGURE 4: Mean square error of the survival probability in Model B

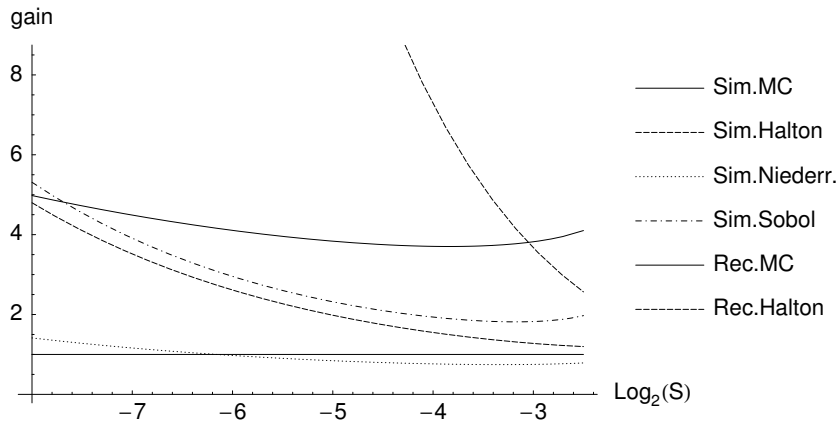


FIGURE 5: Gain for the survival probability in Model B

$b \setminus x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	23.28										
0.1	23.31	24.32									
0.2	23.40	24.44	25.34								
0.3	23.58	24.62	25.56	26.33							
0.4	23.80	24.87	25.87	26.64	27.27						
0.5	24.09	25.21	26.20	27.07	27.75	28.20					
0.6	24.42	25.56	26.64	27.54	28.29	28.86	29.22				
0.7	24.80	26.02	27.10	28.05	28.90	29.54	30.01	30.28			
0.8	25.22	26.50	27.64	28.65	29.50	30.26	30.84	31.27	31.45		
0.9	25.68	27.00	28.17	29.24	30.20	31.00	31.69	32.20	32.56	32.71	
1.0	26.17	27.50	28.75	29.90	30.88	31.76	32.52	33.14	33.61	33.93	34.07

TABLE 4: Exact values of the survival probability in % in Model B

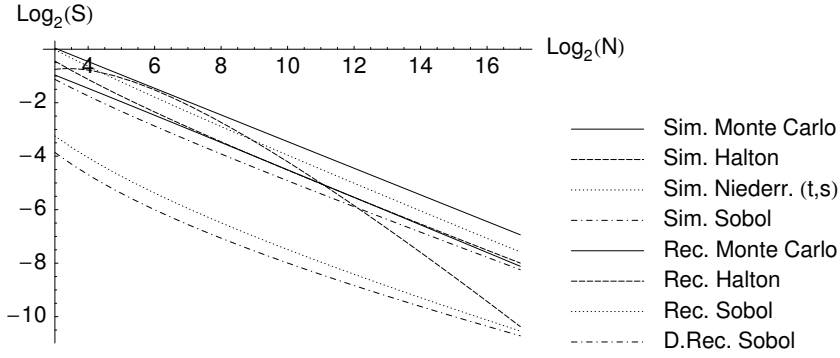


FIGURE 6: Mean square error of the expected dividend payments in Model A

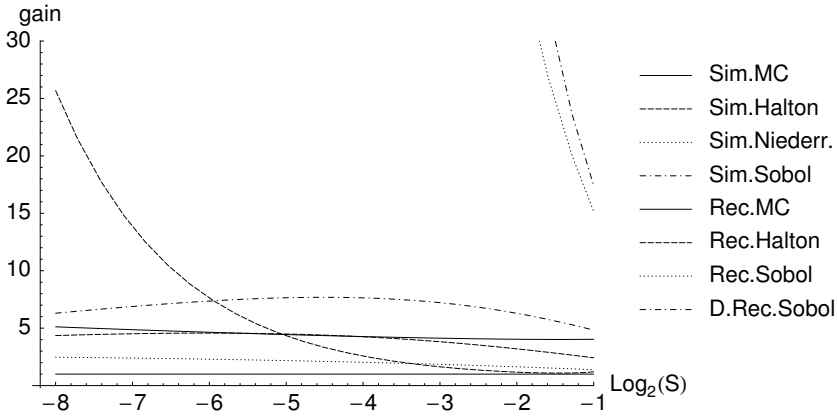


FIGURE 7: Gain for the expected dividend payments in Model A

$b \setminus x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.482										
0.1	1.482	1.592									
0.2	1.481	1.591	1.701								
0.3	1.480	1.590	1.699	1.808							
0.4	1.478	1.588	1.700	1.806	1.913						
0.5	1.476	1.586	1.696	1.805	1.912	2.014					
0.6	1.474	1.583	1.694	1.802	1.908	2.014	2.117				
0.7	1.469	1.582	1.690	1.797	1.904	2.008	2.114	2.215			
0.8	1.466	1.578	1.685	1.793	1.900	2.006	2.110	2.214	2.315		
0.9	1.462	1.572	1.680	1.788	1.894	2.001	2.104	2.208	2.311	2.412	
1.0	1.456	1.565	1.675	1.782	1.886	1.994	2.098	2.201	2.304	2.407	2.506

TABLE 5: Exact values of the expected dividend payments in Model A

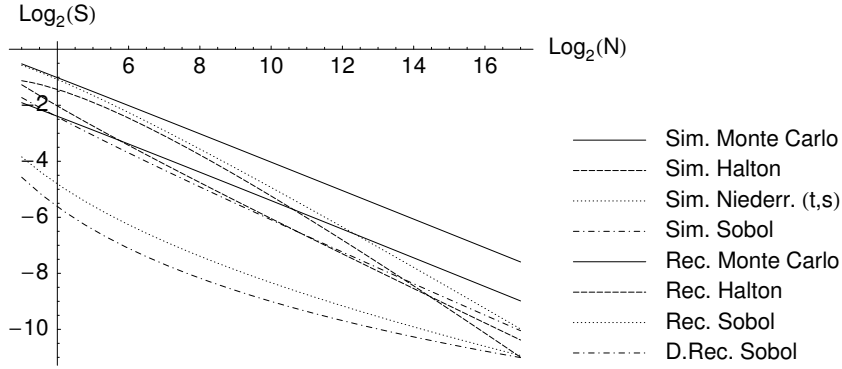


FIGURE 8: Mean square error of the expected dividend payments in Model B

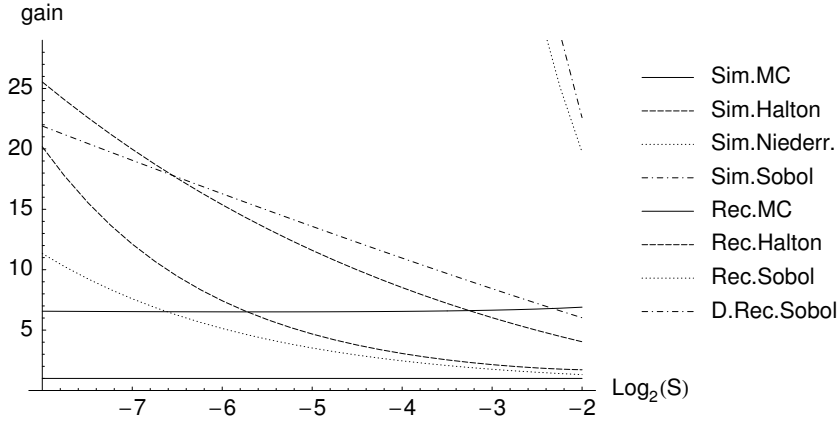


FIGURE 9: Gain for the expected dividend payments in Model B

$b \setminus x$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.045										
0.1	1.045	1.136									
0.2	1.041	1.132	1.225								
0.3	1.036	1.126	1.218	1.312							
0.4	1.028	1.118	1.211	1.302	1.397						
0.5	1.019	1.108	1.198	1.291	1.384	1.479					
0.6	1.007	1.095	1.185	1.276	1.368	1.463	1.559				
0.7	0.993	1.081	1.169	1.258	1.350	1.442	1.536	1.634			
0.8	0.977	1.064	1.151	1.239	1.328	1.420	1.513	1.608	1.706		
0.9	0.960	1.045	1.130	1.217	1.306	1.395	1.486	1.579	1.674	1.773	
1.0	0.940	1.023	1.108	1.193	1.278	1.367	1.457	1.548	1.641	1.737	1.836

TABLE 6: Exact values of the expected dividend payments in Model B

		Monte Carlo	Halton	Niederr. (t,s)	Sobol
Simulation	$S$	0.007141	0.005095	0.006126	0.004136
	$\ \Delta\ _\infty$	0.018727 (163.16 s)	0.010935 (149.58 s)	0.009418 (281.61 s)	0.006817 (150.09 s)
Recursive	$S$	0.004046	0.000755	0.001083	0.000755
	$\ \Delta\ _\infty$	0.012431 (507.3 s)	0.001758 (494.72 s)	0.002598 (494.62 s)	0.001786 (495.04 s)
Double recursive	$S$	0.004309	0.00078	0.000871	0.001054
	$\ \Delta\ _\infty$	0.008598 (3914.71 s)	0.001811 (1761.22 s)	0.002389 (1761.44 s)	0.002432 (3910.15 s)

TABLE 7: Errors for the expected dividend payments in Model A

		Monte Carlo	Halton	Niederr. (t,s)	Sobol
Simulation	$S$	0.004778	0.000855	0.001262	0.000958
	$\ \Delta\ _\infty$	0.010684 (99.71 s)	0.002495 (86.92 s)	0.002914 (87.21 s)	0.002464 (86.91 s)
Recursive	$S$	0.002134	0.000607	0.000479	0.000497
	$\ \Delta\ _\infty$	0.005386 (149.74 s)	0.001762 (136.98 s)	0.001526 (136.8 s)	0.00161 (136.7 s)
Double recursive	$S$	0.002207	0.000709	0.000636	0.000721
	$\ \Delta\ _\infty$	0.005466 (331.1 s)	0.001953 (325.63 s)	0.001843 (325.32 s)	0.002008 (324.69 s)

TABLE 8: Errors for the expected dividend payments in Model B

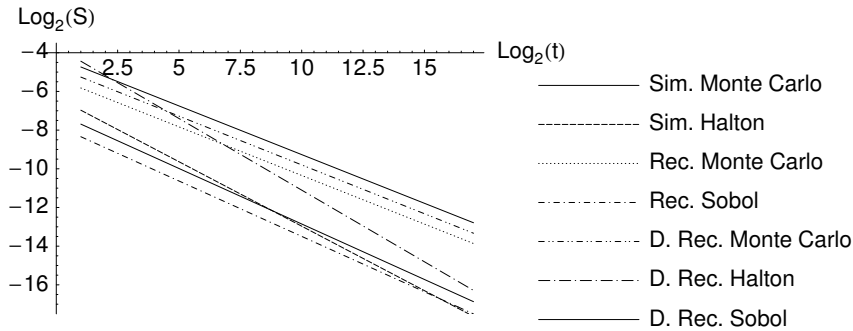


FIGURE 10: Mean square error of the expected dividend payments in Model B, compared with respect to calculation time

## References

- [1] H. Albrecher and R. Tichy. On the convergence of a solution procedure for a risk model with gamma-distributed claims. *Schweiz. Aktuarver. Mitt.*, (2):115–127, 2000.
- [2] A. Alegre, M. Claramunt, and M. Marmol. Dividend policy and ruin probability. *Proceedings of the Third Int. Congress on Insurance: Mathematics & Economics, London*, 1999.
- [3] F. Allen, A. Bernardo, and I. Welch. A theory of dividends based on tax clienteles. *Wharton School Working Paper 15-98, Pennsylvania*, 1999.
- [4] S. Asmussen. *Ruin probabilities*. World Scientific, Singapore, 2000.
- [5] R. Bali and A. Hovakimian. Taxes and stock returns. *Journal of Financial Education*, 25:14–20, 1999.
- [6] P. Boogaert, F. Delbaen, and J. Haezendonck. Macro-economic influences on the crossing of dividend barriers. *Scandinavian Actuarial Journal*, pages 231–245, 1988.
- [7] H. Bühlmann. *Mathematical Methods in Risk Theory*. Springer, New York, Berlin, Heidelberg, Tokyo, 1970.
- [8] R. Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica*, pages 1–46, 1998.
- [9] B. de Finetti. Su un'impostazione alternativa della teoria collettiva del rischio. *Transactions of the 15th Int. Congress of Actuaries*, 2:433–443, 1957.
- [10] M. Drmota and R. Tichy. *Sequences, Discrepancies and Applications*, volume 1651 of *Lecture Notes in Mathematics*. Springer, New York, Berlin, Heidelberg, Tokyo, 1997.
- [11] H. U. Gerber. Martingales in risk theory. *Mitteilungen der Schweizer Vereinigung der Versicherungsmathematiker*, pages 205–216, 1973.
- [12] H. U. Gerber. The dilemma between dividends and safety and a generalization of the Lundberg-Cramér formulas. *Scandinavian Actuarial Journal*, pages 46–57, 1974.
- [13] H. U. Gerber. *An Introduction to Mathematical Risk Theory*. Huebner Foundation Monograph 8, Homewood, Ill, 1979.
- [14] H. U. Gerber. On the probability of ruin in the presence of a linear dividend barrier. *Scandinavian Actuarial Journal*, pages 105–115, 1981.
- [15] H. U. Gerber and E. Shiu. On the time value of ruin. *North American Actuarial Journal*, 2(1):48–72, 1998.
- [16] J. H. Halton. On the efficiency of certain quasi-random sequences of points in evaluating multidimensional integrals. *Numer. Math.*, 2:84–90, 1960.
- [17] J. Irbäck. Asymptotic theory for a risk process with a high dividend barrier. *Licentiate thesis, Royal Institute of Technology Stockholm*, 2000.

- [18] H. Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods*. Society for industrial and applied mathematics, Philadelphia, Pennsylvania, 1992.
- [19] J. Paulsen and H. Gjessing. Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance: Mathematics and Economics*, 20:215–223, 1997.
- [20] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes in C*. Cambridge University Press, Cambridge, 1992.
- [21] C.-O. Segerdahl. On some distributions in time connected with the collective theory of risk. *Skand. Aktuarietidskr.*, pages 167–192, 1970.
- [22] T. Siegl and R. F. Tichy. Lösungsmethoden eines Risikomodells bei exponentiell fallender Schadensverteilung. *Mitteilungen der Schweizer Vereinigung der Versicherungsmathematiker*, 1:85–118, 1996.
- [23] I. M. Sobol'. On the distribution of points in a cube and the approximate evaluation of integrals. *USSR Comput. Math. Math. Phys.*, 7:86–112, 1967.
- [24] O. Thorin. Probabilities of ruin. *Skandinavian Actuarial Journal*, pages 65–102, 1982.
- [25] R. F. Tichy. Über eine Zahlentheoretische Methode zur numerischen Integration und zur Behandlung von Integralgleichungen. *Sitzungsberichte der Österreichischen Akademie der Wissenschaften*, 193, 1984.
- [26] R. F. Tichy. Bemerkung zu einem versicherungsmathematischen Modell. *Mitt. Verein. Schweiz. Versicherungsmath.*, (2):237–241, 1987.

H. ALBRECHER AND R. KAINHOFER  
 Department of Mathematics  
 Graz University of Technology  
 Steyrergasse 30  
 8010 Graz  
 Austria